Distinction and the Foundations of Arithmetic

Thomas J. McFarlane
thomasmc@stanfordalumni.org
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ABSTRACT

An arithmetic of rational numbers is developed organically from the idea of distinction. By explicitly representing distinctions as they unfold naturally, a hierarchy of increasingly complex arithmetic systems are shown to grow out of the same root. Conflating some distinctions produces the binary arithmetic of *Laws of Form* by Spencer-Brown, while conflating other distinctions produces the boundary arithmetic of Jeffrey James. The present paper thus provides a broader contextual framework within which these systems can be related and seen as different branches in the same tree growing from the seed of a primordial distinction.

INTRODUCTION

What are numbers, really? If *everything is made of number*, as Pythagoras declared, then it is necessary to understand the nature of number to understand the essence of all things.

In one sense, numbers are to mathematics as the atom is to classical physics. They may be seen as a fundamental irreducible basis for everything else. When we learn to count, for example, we learn one of the most basic properties of numbers: that they form an ordered sequence. This property was formalized when the Italian mathematician Peano showed that, to generate all whole numbers, all we need is one number and the ability to create a new number from an existing number: From 0 we can create 1, and from 1 we can create 2, and then 3, and so on. Thus, all whole numbers can be generated from zero using a single principle. For centuries, mathematicians looked no deeper than this. They were satisfied with reducing number to this basic principle, much as physicists before quantum theory were satisfied with the assumption that all matter is reducible to indivisible atoms. When physicists penetrated to the subatomic level, an entirely different world opened up, completely transforming our understanding of the nature of matter. Similarly, to understand the nature of number, we need to look more deeply.

When the Pythagoreans said that everything is made of number, they meant everything is made of the whole numbers: 1, 2, 3, etc. Based on this assumption, it must be possible to make all other kinds of numbers from whole numbers. For example, a rational number (i.e., a fraction) can be made by dividing one whole number by another. As the Pythagoreans realized, however, irrational numbers (e.g., the square root of two) are impossible to generate in any simple way from whole numbers. Naturally, this was a big problem for the Pythagorean principle that everything is made of number. How can irrational numbers be defined?

This problem stood unsolved for over two thousand years until mathematicians Dedekind and Cantor defined irrational numbers in terms of infinite sets. This revolutionized the conception of number, and the foundations of all of mathematics, just as the discovery of sub-atomic particles and quantum mechanics revolutionized the foundations of physics. For the first time, mathematicians were able to understand numbers in terms of a more fundamental "sub-numerical" conception: the idea of a set. For example, natural numbers may be constructed from sets by first constructing the empty set, {}, which is the set containing nothing. We can then construct the set containing the empty set, {{}}, and the set containing those two sets, {{},{{}}}, and so on. Thus, the same pattern that Peano devised is reproduced from the more fundamental notion of set. Building upon these sets, one may construct sets that correspond to the rational numbers, irrational numbers, and so on. The essence of number (and most any mathematical idea at all) can be understood in terms of the idea of set. The modern Pythagorean thus declares that *everything is made of sets*.

What, then, is a set? It turns out that this notion is at once profound and paradoxical. According to Cantor, "a set is a Many that can be thought of as a One." If we permit any "Many" to be a set, however, we discover the Russell paradox. Collect together all the sets that are *not* members of themselves. This set must either be a member of itself or not. If it is a member of itself, then it is one of the sets that are not members of itself, which is a contradiction. If it is not a member of itself, then it isn't actually the set of all sets that are members of itself, which is also a contradiction. Thus, the collection of all sets that are not members of themselves is an example of a Many that cannot be made into a One. Thus, when allowed to make anything into a set, the notion of set leads to contradictions and exhausts its capacity to function consistently. At a certain point, sets cannot take a Many and make them into a One without contradiction. In other words, when collections get "too large" they exceed the limits of our capacity to define them, and we are lead to the limits of consistent thought. In some sense, the notion of a set has paradox implicit within it.

We can gain more insight into the idea of set by examining its most primitive instance: the empty set, { }, the set that contains nothing. Because it contains nothing, the empty set is the bare idea of making a set, of creating something. The empty set is a thing, but what it contains is nothing. This "nothing," however, should not be mistaken for some kind of subtle "something."

We have not taken a "thing" called nothing and made a set out of it. We have made a set out of ineffable no-thing-ness.

As we have seen from the Russell paradox and a consideration of the empty set, we run into the ineffable at both the simplest root of all sets as well as at the farthest reaches of infinite sets. Prior to every set and beyond every set is the ineffable. There thus appears a coincidence of opposites: the ineffable is both everything and nothing. Let us symbolize this ineffable everything-and-nothing by A. The ineffability of A, the inability of concepts and distinctions to capture it, may be expressed by writing $A=\{A\}$. In other words, with respect to A, the operation of making a set does not create a difference, does not make a distinction. Or, A is invariant under the operation of making a distinction. The equation $A=\{A\}$ may also symbolize the coincidence of opposites: A seen as the ineffable infinite everything beyond all sets coincides with A seen as the ineffable empty nothing prior to all sets.

The conventional notion of set, on the other hand, is based on the premise that the operation of making a set makes a difference. This may be represented as $\{A\}\neq A$. In other words, making a set now makes a difference and creates something distinct. This may be interpreted as the first set, the empty set, and all other sets are built up from it. Thus, all sets are derived from the primary distinction between nothing and everything. Set theory thus supposes that $\{A\}\neq A$, rather than $\{A\}=A$. It supposes that A can actually be separated from itself. Set theory, and all of mathematics, arises from this fundamental assumption that a set can create a difference.

Because every set presupposes a distinction, the idea of distinction is even more fundamental and more general than the idea of set. Every set is defined or created by making a distinction between what is contained in the set, and what is not. So, sets are made of distinction. The Pythagorean maxim then becomes: *everything is made of distinction*. In fact, this is not hard to see, once one notices it. Every thought directs attention to one portion of the world as distinct from another. Every idea contains an implicit distinction between that which it indicates and that which it does not indicate. In fact, we can get no more fundamental than to make a distinction, for whenever we have any thing at all we have at least one distinction: the distinction between that thing and no-thing. There is something precisely because there is distinction — thus, all things are created by distinctions, and exist as distinctions.

RECOVERING LOGIC AND ARITHMETIC FROM DISTINCTIONS

Taking distinction as fundamental, let us now attempt to see exactly how mathematics begins to emerge naturally from distinction. We have already seen that sets are really nothing other than distinctions. And, just as we can make numbers by repeatedly creating sets, we should be able to make numbers from repeated distinctions. Indeed, in his book *Laws of Form*[1] G. Spencer-Brown develops a binary arithmetic based upon transformations of distinctions. What is most impressive about his system is that both the numbers and the transformations of them are distinctions. In other words, the arithmetical operations and the numbers they operate on are both distinctions. To give a sense of the elegance of his system, we will take a brief look at some of its main features.

A distinction in Spencer-Brown's arithmetic is represented by a *cross*,

$$\urcorner$$

An *expression* in the arithmetic is formed from a collection of crosses which are nested within or juxtaposed to each other. In addition, an empty space, , is a valid expression whose value in the arithmetic is conventionally called the *void*. Expressions in the arithmetic are transformed into each other through the application of two fundamental axioms, the axiom of *condensation* and the axiom of *cancellation*:

$$\urcorner\urcorner \quad = \urcorner \qquad \text{condensation}$$

$$\overline{\,\overline{}|\,}| \quad = \qquad \text{cancellation}$$

With these two axioms, any expression is equivalent to either the cross or the void. These are also referred to as the *marked state* and the *unmarked state*. Remarkably, this arithmetic system naturally gives rise to an algebra that is equivalent to the laws of Boolean logic. Thus, all the basic laws of logic can be seen as nothing more than distinctions, and unfolding naturally from the single idea of making a distinction.

The arithmetic in *Laws of Form* is limited to binary arithmetic. In an unpublished article[2], however, Spencer-Brown proposed an arithmetic of natural numbers based upon distinction by limiting the axiom of condensation, so that

$$\urcorner\urcorner \quad \neq \urcorner \quad .$$

In his proposed arithmetic, the natural numbers 1, 2, and 3 correspond to the distinct expressions

$$\urcorner \qquad \urcorner\urcorner \qquad \urcorner\urcorner\urcorner$$

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[1]G. Spencer-Brown, *Laws of Form*, Bantam, New York, 1969.

[2]G. Spencer-Brown, "An Algebra for the Natural Numbers", unpublished article, 1961.

In general, a number *n* corresponds to the juxtaposition of *n* crosses, sometimes denoted

$$\urcorner_n$$

Using this notation, we may write, for example,

$$3 = \urcorner_3 \qquad = \urcorner\urcorner\urcorner$$

The operation of addition is naturally defined by juxtaposing two expressions in the same space:

$$a + b = \urcorner_a \urcorner_b \qquad = \urcorner_{a+b} \qquad .$$

Spencer-Brown defines the operation of multiplication by

$$p * q = \overline{\overline{p|}\;\overline{q|}}\Big|$$

This expression for a product, however, has no obvious interpretation as a natural number unless it can be transformed to an expression that is a sequence of juxtaposed crosses. This brings us to the question of what transformations of expressions are allowed in this number system. In *Laws of Form*, Spencer-Brown developed an *algebra* corresponding to his binary arithmetic; the algebra is based on two algebraic axioms called *position*, and *transposition*:

$$\overline{\overline{p|}\;p}\Big| \qquad = \qquad , \qquad\qquad \text{position}$$

$$\overline{\overline{pr|}\;\overline{qr|}}\Big| \qquad = \overline{\overline{p|}\;\overline{q|}}\Big| \qquad r. \qquad\qquad \text{transposition}$$

Just as Spencer-Brown based the arithmetic of natural numbers on cancellation alone, so he based its algebra on transposition alone. (He also, however, makes use of an algebraic generalization of cancellation which is derived from both these axioms, and not just transposition.) Using transposition, non-numerical multiplicative expressions may be simplified to yield a numerical result. (The expression for 0×0, however, is not reducible, so is considered distinct from zero.) It should also be noted that unrestricted use of transposition implies distribution of addition over multiplication, i.e., $(p+r) \times (q+r) = (p+q) \times r$, which is false in conventional arithmetic. We must therefore introduce an *ad hoc* limitation to the application of transposition so that we do not derive false statements in arithmetic. Kauffman and Engstrom[3]

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[3]Jack Engstrom, "Natural Numbers Derived from Laws of Form", Master of Science Thesis, Maharishi International University, 1994.

suggest some ways to solve some of these problems with Spencer-Brown's system by the introduction of a secondary implicit distinction between additive and multiplicative spaces. There are also other questions regarding the justification for the foundations of this system. What, for example, is the justification for adopting the transposition law, which was proved on the basis of the original axioms, one of which is now altered? What is the justification for limiting the application of this law? And what is the justification for adopting some of the other algebraic results of the original system, while rejecting others? As it stands, the foundations of this arithmetic system appear *ad hoc*. It would be desirable to have a more intuitive and continuous link with the system upon which it is based. In addition, it is not clear how this system may be extended beyond natural numbers to obtain an arithmetic of integers or rationals.

Jeffrey James[4] has also developed an arithmetic based on distinctions (or, as he calls them, boundaries). His arithmetic system includes integers, rational numbers, and even some real and imaginary numbers. James assumes as given several different types of distinctions and bases their transformations on a set of axioms. Rather than using Spencer-Brown's cross notation, James represents a distinction by one of three different types of bracket pairs: round brackets, ( ), square brackets, [ ], and pointed brackets, < >. These bracket pairs are combined to form expressions in his calculus. The virtue of James' system is that he obtains a surprisingly powerful arithmetic based upon these three distinctions and his rules for their transformation. The drawback is that the transformative rules and multiple types of distinctions are introduced without providing any intuitive basis for them. Nor are they all consistent with, or linked in an intuitive way to, the rules and distinctions of Spencer-Brown.

INTRODUCTION TO THE PRESENT SYSTEM

Much of the elegance of Spencer-Brown's system is the organic development of rules and axioms out of the simple idea of distinction, providing his system with an intuitive basis, making it more than an arbitrary set of axioms and definitions. It would be very desirable to develop an arithmetic of integers and rationals through an organic and natural progression from the idea of distinction, in much the same way Spencer-Brown developed his system.

This paper provides a general context for understanding how distinction gives rise to successive arithmetic systems of increasing complexity. At the core is the fundamental idea of

[4]Jeffrey M. James, "A Calculus of Number Based on Spatial Forms", Master of Science Thesis, University of Washington, 1993.

distinction. At this basic level, the arithmetic is trivial and quite uninteresting since there is not yet sufficient complexity to provide any structure. The first non-trivial levels provide simple arithmetics, such as the binary arithmetic described by Spencer-Brown in *Laws of Form*. With additional complexity, there then arises an arithmetic of addition and multiplication of positive integers, as developed by Spencer-Brown, Engstrom and Kauffman. The arithmetic of rationals, which includes the multiplicative and additive inverses, as developed by James, arises through the introduction of additional distinction and structure. The contribution of this paper is to place all these systems in the context of a coherent whole, showing how they are all naturally connected and can be traced back to a single distinction.

The Ground

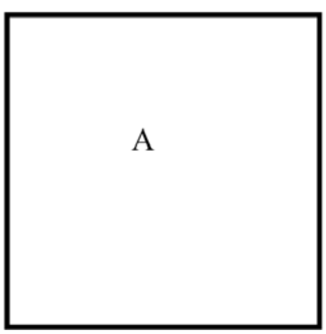
First, we acknowledge the implicit ineffable Absolute prior to all distinction. This statement is inherently paradoxical because it is based upon a distinction between what is prior to distinction and what is not. But if the Absolute is prior to every distinction, it is prior even to the distinction between distinction and non-distinction. Because distinction is the basis of logic, the Absolute is prior to logic. Because distinction is the basis of all language, the Absolute is prior to words. The Absolute it is not opposed to or exclusive of anything, because it is not ultimately distinct from anything, including distinction. The Absolute encompasses and comprehends all distinction, language, and logic, and everything that is outside of distinction, language and logic. At this ultimate level of infinite comprehension, every distinction, every word, both indicates the Absolute and is itself not distinct from the Absolute. Everything is thus an indication of the Absolute, and is the Absolute. So, through things, the Absolute indicates itself, refers to itself. There is nothing apart from the Absolute, so form and formlessness are indistinguishable from each other and from the Absolute.

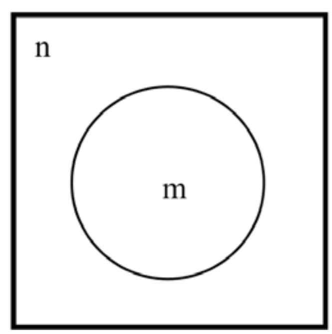
This insight, when expressed in the language of logic and its distinctions, appears as contradiction and self-reference. We symbolize this as <A>=A, meaning A seen as “thing” is identical to A seen as “nothing.” Whether we say something or nothing, we indicate A. This yields a trivial arithmetic of unity. All forms, all signs, all symbols, as well as the absence of these, are all identical with each other and with the Absolute. This arithmetic of unity has only one value, expressed by (and identical to) all the infinite forms and their formless essence. It is thus the ultimate in simplicity.

The Primordial Distinction

Insofar as everything is identical with the Absolute, there is nothing that has any ultimately distinguishable identity. Yet, although A=<A> expresses the identity of A as something and A as nothing, it also expresses the possibility of imagining two aspects of A. Now we suppose that the essential identity of these two aspects of A is forgotten or ignored. Then, suddenly, something appears to exist as distinct from nothing. Form is severed from formlessness, A≠<A>. Thus, distinction arises as the result of an ignorance of identity. The original A is eclipsed by its two incomplete aspects, now appearing as if they were real and independent of each other. The meaning of the symbol A has now shifted in a subtle but significant way. The original A equals <A>, while this new A does not equal <A>. In other words, the original A includes both form and formlessness, while the new A excludes form and includes only formlessness. Because formlessness exclusive of form should not be confused with the original A, let us represent these two distinguished aspects of A as m and n. We interpret m as the something created by the distinction, and n as the aspect remaining of A after something was distinguished from it. Thus, we can write m=<n> and n=<m>. In other words, m is distinct from n, and n is distinct from m.

We can symbolize the above process in a concrete way as follows. Take a blank sheet of paper and draw a circle in the space of the page. Now the circle allows us to see two aspects of the blank space of the page: a space inside the circle and a space outside the circle. To distinguish these two aspects of the total space, we mark these two spaces with m and n. The inside space is a symbol for m, and the outside space is a symbol for n, and the whole page is a symbol for A.





The equation m=<n> means that the space m is on the other side of the distinction from n, and n=<m> means that n is on the other side of the distinction from m. It can also mean that an indication of m together with an indication to cross the distinction is the same as an indication of n. The distinction < > can thus be seen as an instruction to cross from the space indicated inside the brackets to the space on the other side of the distinction. Now, since crossing the distinction does not take you out of the page, it is still true that A=<A>. If we use a blank space instead of A to symbolize this total space of the page, then this becomes = < >. Since m and n are distinct from each other, we also have m≠n. And since m and n are only parts of the space, and not the whole space, we have m≠ , and n≠ .

It should be emphasized that the above development differs in an important respect from *Laws of Form*. We have distinguished the unmarked state, n, from the entire space of the page, A. In contrast, *Laws of Form* confuses the space n with A. In other words, it has n= . And since m=<n>, it follows that m=<n>=< >. Thus, in *Laws of Form* the state m and the distinction < > are not separate. In *Laws of Form*, distinction is confused with the very thing that distinction creates, and the Absolute is confused with the opposite of distinction. In the present system, on the other hand, m≠< > and n≠ . Another difference is that *Laws of Form* has < >≠ , while we have < >= . These differences illustrate how distinct systems can organically grow out of the ineffable Absolute, as well as how *Laws of Form* can be seen as arising through the process of identifying or confusing distinct aspects of the present system. Tracing how each system is rooted in the Absolute reveals how they are related to each other.

In summary, if we write < > or , we indicate A. If we write n or m, we indicate one or the other of the two distinguished aspects of A. If we indicate <m>, we indicate n. And if we indicate <n>, we indicate m.

Multiple indications

Now notice that the indications <m> and <n> are actually compound indications. Two indications are taken together as one. The indication<m> instructs us to indicate m and cross the distinction, i.e., indicate what is distinct from m. To perform this compound indication, we need to be able to combine indications. If we limit ourselves to just the simplest of indications, we have just m, n, and < >. To interpret compound indications, we need to take two indications and consider them as one indication. More generally, we need to be able to indicate multiple indications and take them as a single indication (we should be reminded here of Cantor's definition of a set). This capacity to make a set thus emerges at the very next step after the power to distinguish, and allows us to regard a collection of indications as a single indication.

Since n and m represent complementary aspects of A, if we want to indicate A, we can simply indicate both n and m together. Thus, it is natural to have nm= =< >. So the values of <m>, <n> and mn are defined. But what about the expressions mm and nn? Indicating m twice obviously does not indicate A, or n. Nor is it exactly the same as a single indication of m because it is a double indication of m. Similarly, nn is a double indication of n. Using the capacity to distinguish two indications from one, we have mm as indicating something new. Like m, it still indicates the state m, but it indicates it twice, and not just once. We say the same for nn. Thus, at the level of multiplicity of indications it is natural to consider mm as distinct from m, and nn as distinct from n. In another sense, however, insofar as mm is two indications of m, it is also indicating m, albeit twice instead of just once. With this in mind, consider now the expression <mm>, which means indicate m twice and cross. Since indicating m once and cross is equivalent to indicating n, we may naturally consider <mm> as equivalent to indicating n twice. Thus, <mm>=nn=<m><m>. Similarly, <nn>=mm=<n><n>. This is also consistent with the fact that <nm>=<A>=A=nm=<n><m>. Thus, for any two single indications x and y, we have <xy>=<x><y>.

Next consider the expressions <<m>> and <<n>>. Because we know that n=<m> and m=<n>, it follows that n=<<n>> and m=<<m>>. Moreover, < >=   implies that << >>=   . So for any single indication x, we have x=<<x>>.

Also note that because nm=   , it follows that n<n>=   , and m<m>=   . Moreover, < >=   . So, for any single indication x, we have <x>x=   .

We thus have an intuitive basis for the following algebraic laws:

<xy>=<x><y>   x=<<x>>   <x>x=   .

And the basic arithmetic laws are:

n=<m>   m=<n>   nm=< >=   .

As an aside, note that we can make a simple closed arithmetic system by confusing mm and nn with A, i.e., mm=nn=   . We then have a system of three values: m, n, and   , and when any two values are combined, we obtain again one of the three values. (This system is isomorphic to $\mathbf{Z}_2$, with < > interpreted as negation, and xy interpreted as x+y.)

Now if, on the other hand, we do not add mm=nn=   , then these double indications act as distinct entities, new kinds of elements. We then get a system of five values: nn, n,   , m, mm. (If we add the law mmm=nnn=   , then this system is isomorphic to $\mathbf{Z}_3$.) We can continue this process and obtain, for any N, an arithmetic system of 2N+1 values. (This system is isomorphic to $\mathbf{Z}_N$.) The system will have the following basic arithmetic and algebraic laws

<xy>=<x><y>   x=<<x>>   <x>x=

n=<m>   m=<n>   nm=< >=   n...n=m...m=

The last rule states that N repetitions of n (or m) indicates nothing. If N is larger than any indication we use, however, this rule is never used. In other words, because we can always take N to be larger than any indication we use, we can drop this last rule and then the system effectively describes the infinite set of integers under addition and subtraction.

Multiplication

In the previous section, we created integers from collections of indications: m, mm, mmm, mmmm, etc. Even though, in one sense, these expressions all indicate m, we distinguish them by the number of times m is indicated. Now observe that the counting of m's is no different than the counting of n's. We use the same counting operation on both indications. Just as the operator < > is an instruction to cross the distinction from a given state, regardless of which state, this counting operator repeats indications, regardless of what those indications are. These counting operators do not by themselves indicate m or n. We have abstracted from particular indications of states here, and are now representing the number of repetitions alone, without any particular state being indicated.

The multiple indications m, mm, mmm, mmmm, etc. arise by application of counting operators 1, 2, 3, 4, etc. to the single indication m. More generally, when the operator 1 is applied to an expression, it means to indicate that expression once. When the operator 2 is applied to an expression, it means to indicate that expression twice. Thus, these counting operators act as multipliers when applied to expressions (which are, in turn, operators that indicate a state). Because the application of these repetition operators to an expression (i.e., multiplication) is distinct from juxtaposition of indications (i.e., addition), we distinguish this new level of abstraction from the prior level with brackets [ ] to indicate this distinct type of juxtaposition applies to the elements in brackets. Thus we write [1][n]=[n], [2][n]=[nn], [3][n]=[nnn], [4][n]=[nnnn], etc. and similarly, [1][m]=[m], [2][m]=[mm], [3][m]=[mmm], [4][m]=[mmmm], etc. Now we can write [3][2][m]=[3][mm]=[mm mm mm].

To combine both addition and multiplication in the same expression, we actually need a second distinction to provide uniqueness. For example, since [x][y]ab[p][q] = [x][y][p][q]ab can be evaluated in different ways, we need a second distinction ( ) to indicate how the bracketed elements [ ] are to be grouped together, otherwise the meaning is ambiguous. Thus, ([x][y])ab([p][q]) ≠ ([x][y][p][q])ab. The brackets [ ] thus shift an expression into multiplicative space, while the brackets ( ) shift back to additive space. In general, we can see that ([x])=x. These distinctions thus take us back and forth across the distinction dividing two levels of abstraction (i.e., between additive and multiplicative spaces).

To recover the boundary arithmetic of Jeff James, we identify m=( ). From this, we see immediately that ([2][m])=([mm])=mm=( )( ). So, 2=([2])=([2][( )])=([( )( )])=( )( ). Thus, the operator 2 is now identified with two indications of m. The brackets determine whether 2 is to be taken as an operator to repeat any indication twice, or as a double indication of m. In other words, by clearly distinguishing the multiplicative and additive spaces, we are free to conflate the operator 2 with the operand mm.

We now make the correspondence with boundary arithmetic explicit. First note that it follows immediately from m=( ) that n=<( )>.

Now consider distributivity. In general, r(x+y)=rx+ry translates to ([r][xy])=([r][x])([r][y]), where r, x, y are numbers. In particular, if x=y=( ), then ([r][( )( )])=([r][( )])([r][( )])=([r])([r]) =r+r, i.e., 2r=r+r. Note: we can also write distributivity as (rx)(ry)=([(x)(y)]r). We may take either form as our axiom.

Since taking a number no times is nothing, we require that ([0][p])=0, or ([ ][p])= . So if p=0, then ([ ][ ])= . Since taking a number once is the number, ([1][p])=p=([p]), so [1]=[( )]= . This is distinct from ([ ])= , which we already have. Note we can summarize this as p=([p]) and [ ]=[ ][p].

Now observe that 3^2 = 3×3 = ([3] [3]) = ([( )( )( )] [( )( )( )]) = (([[( )( )( )] [( )( )( )]]))

= (([([[( )( )( )]]) ([[( )( )( )]])])) = (([[( )( )( )]] [( )( )])) = (([[3]][2]))

In general, a^b=(([[a]][b])).

We thus have all the axioms needed to reproduce the system of Jeff James, having developed them organically from the idea of distinction.

AXIOMS:

I. <p>p=

II. (rx)(ry)=([(x)(y)]r) or (r[xy])=(r[x])(r[y])

III. ([x])=[(x)]=x

IV. x[ ]=[ ]

Compare I and II to the axioms of *Laws of Form*:

$$\overline{\overline{p|}\ p|} = \quad , \qquad \text{position}$$

$$\overline{\overline{pr|}\ \overline{qr|}|} = \overline{\overline{p|}\ \overline{q|}|}\ r. \qquad \text{transposition}$$

From axiom I it follows that < >= . So, by applying axiom I twice, <<p>p>=< >= , thus obtaining the form of the position axiom of *Laws of Form*. Note also that axiom I implies that <<x>>=x and that <xy>=<x><y>. Now observe that by axioms II and III, [(pr)(qr)]=[([(x)(y)]r)]=[(x)(y)]r, thus obtaining the form of the transposition axiom of *Laws of*

Form. Axioms III and IV reflect the form of two consequences (which Spencer-Brown calls C1 and C3) that are derived from position and transposition axioms. Thus, *Laws of Form* can be seen as an arithmetic system that arises by conflating the three different types of distinctions.

Finally, we can show how the < > distinction interacts with the [ ] and ( ) distinctions: <(a[b])> = <(a[b])> ([ ]) = <(a[b])> (a[ ]) = <(a[b])> (a[b<b>]) = <(a[b])> (a[([b])([<b>])]) = <(a[b])> (a[b]) (a[<b>]) = (a[<b>]). Thus, < > can be moved outside or inside across the two complementary boundaries: <(a[b])> = (a[<b>]).

We then have the following correspondences:

0, 1, 2	, ( ), ( )( )
–0, –1, –2	< >, <( )>, <( )( )>
–0=0	< > =
a+b	ab
a–b	a<b>
a×b	([a][b])
0×a=0	([ ][a])=([ ])=
0×0=0	([ ][ ])=([ ])=
-p+p=0	<p>p=
rx+ry=r(x+y)	(rx)(ry)=([(x)(y)]r)
–(–(x))=x	<<x>>=x
–(x+y)=–x+–y	<xy>=<x><y>
–(ab)=a(–b)	<(a[b])> = (a[<b>])
a^b	(([[a]][b]))
$x^0=1$	(([[x]][ ])) = ()
$0^0=1$	(([[ ]][ ])) = ()
$0^1=0$	(([[ ]][()])) =
$1/a = a^{-1}$	(<[a]>) = (([[a]][<()>]))
a/b	([a]<[b]>)
(1/x)(1/y) = 1/xy	(<[x]><[y]>)= (<[x][y]>)

x/x = 1	([x]<[x]>) = ()
1/(1/x) = x	(<[(<[x]>)]>) = x
a/b + c/d = (ad+cb)/bd	([a]<[b]>) ([c]<[d]>) = ([([a][d])([c][b])]<[b][d]>)
0/x=0	([ ]<[x]>) = ([ ]) =
1/0	(<[ ]>) does not reduce (could define this as infinity?)
0/0	Ambiguous. Can reduce to 0 or to 1.
$x^{1/y}$	(([[x]]<[y]>))
√2	(([[()()]]<[()()]>))
$i=(-1)^{1/2}$	(([[<()>]]<[()()]>))

The exponential

James presents the following correspondences:

e^x	(x)
ln x	[x]

We can perhaps provide a natural justification for these interpretations. Consider a function F that is defined by x → (x). In particular, xy → (xy) and Now, because x=[(x)] and y=[(y)], we have (xy)=([(x)][(y)]). Thus, by the definition of multiplication, F(x+y)=F(x)F(y). From this it follows that F(x)=F(0+x)=F(0)F(x), so F(0)=1. In other words, ( )=( ). Moreover, 1=F(0)=F(x–x)=F(x)F(–x), so F(–x)=1/F(x). In other words, (<x>)=(<[(x)]>). In addition, F(nx)=F(x+...+x)=$F(x)^n$, and in particular, F(n)=$F(1)^n$. In other words, (([n][x]))=(([n][[(x)]])) and (n)=(([[(( ))]][n])). We can also see that F(nm)=$F(1)^{nm}$, i.e., (([n][m]))=(([[(())]][n][m])). So, the function F defined by x → (x) has the following properties:

F(0)=1

F(x+y)=F(x)F(y)

F(–x)=1/F(x)

F(n)=$F(1)^n$

F(nx)=$F(x)^n$

We can extend the function F to rational arguments by observing that F(m)=$F(m/n)^n$ and in particular F(1)=$F(1/n)^n$. So F(1/n) is clearly the n^{th} root of F(1). Note, however, that roots are not necessarily unique.

We see that the function F has essentially all the properties of exponentiation with base F(1). Now F(1)=(( )). Thus if we use the correspondence $a^b \Leftrightarrow$ (([[a]][b])) to write $F(1)^x$, we get (([[(( ))]][x]))=(x). Thus (x) is an exponential to an implicit base (( )). The use of base (( )) is not necessary, but is natural because it provides an especially simple form of the exponential.

If we treat this base as an irreducible number, we do not necessarily need to know the value of F(1) in terms of integers. The distinctions ( ) and [ ] represent a switching between multiplicative and additive spaces, and that is just the essence of what the exponential function and its logarithmic inverse does.

Using limits, however, we can extend the function F to the reals and determine a natural value for the base (( )). If we define the real number x as the limit of the sequence of rationals $<x_n>$ as n goes to infinity, then we can define F(x) as the limit of $F(x_n)$. Using a Taylor series and limits we can extend F to the complex numbers. Suppose $F(x)=a_0 + a_1x + a_2x^2 + ...$ From the property F(x+y)=F(x)F(y), it can be shown that $F(x) = 1 + ax + (ax)^2/2! + (ax)^3/3! + ...$ Now if we make the natural choice that F′(x)=F(x), so that a=1, then we have $F(x)= 1 + x + x^2/2! + x^3/3! + ...$, which is the power series for e^x. (The number e can be defined as equal to the limit of $(1+d)^{1/d}$ as d approaches 0, or indirectly by saying the e is the number such that $(e^d-1)/d$ approaches 1 as d approaches 0.) This series is defined for integers, rationals, reals, and complex numbers. One may factor it into real and imaginary parts, and obtain F(ix) = cos x + i sin x, where cos x and sin x are the real and imaginary parts. Of course, this extension to the reals has simply assumed the notion of limit, when it has not yet been developed. It remains an interesting open question to develop real numbers and limits from the present system in an organic way.